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A STRUCTURE THEOREM FOR ORTHOGROUPS

A.H. CLIFFORD
Tulane University, New Orleans, Lousiana, U.S.A.

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An orthodox semigroup is a regular semigroup S such that the set E_S of idempotents of S is a subsemigroup of S. An orthodox semigroup which is a union of groups is called an *orthogroup*. Structure theorems for orthogroups have been given by Fantham [3], G.B. Preston (1961, unpublished; see [1]), Yamada [14], Warne [11], and Petrich [10]. The main purpose of the present paper is to add another (Theorem 1.1 below) to the list.

A full description of every possible orthogroup would, of course, include a description of every possible band (semigroup every element of which is idempotent). This is the case with the theorems of Warne and Petrich, both of which specialize to interesting, though complicated, structure theorems for bands.

The other three theorems are less ambitious, in that a knowledge is assumed of the band E_S of idempotents of the orthogroup S to be constructed. Preston and Yamada simplify matters still further by assuming also that we know the greatest inverse homomorphic image Q_S of S. The point of view here is to begin with a band E and a semilattice of groups Q which are compatible in the sense of having isomorphic structure semilattices, and then proceed to find all "orthodox extensions of E by Q", that is, all orthogroups S such that $E_S \cong E$ and $Q_S \cong Q$. This point of view is adopted in the present paper.

In all of these theorems, product in S is defined in terms of systems of mappings satisfying certain conditions. No procedures are given for constructing such systems, and no over-all perspective of their totality is provided. There does exist, however, a structure theorem for orthogroups due to T.E. Hall which gives us a commanding perspective of all possible orthodox extensions S of a given E by a given Q, and product in S is defined (at least in theory) in a simple and natural manner.

Generalizing Munn's well-known fundamental representation [8] of inverse semigroups, Hall showed in [4] that, for any band E, there exists a greatest fundamental orthodox semigroup $W(E)$ having E as its band of idempotents. In [5], he showed that any orthodox semigroup S is isomorphic to the spined product of Q_S and $W(E_S)$ relative to some homomorphism $\psi: Q_S \to Q_{W(E_S)}$ and the natural homomorphism $W(E_S) \to Q_{W(E_S)}$. (Following [7] and [15], we define the spined product of two semigroups S_1 and S_2 relative to homomorphisms $\psi_1: S_1 \to S_3$ and ψ_2: $S_2 \to S_3$ to be the subsemigroup of their direct product $S_1 \times S_2$ consisting of all pairs (a_1, a_2) such that $a_1\psi_1 = a_2\psi_2$.)

Hall also showed (verbal communication to the author in 1972) that every orthodox semigroup S contains a greatest full suborthogroup K_S ("full" meaning "containing E_S"). Writing $K(E)$ for $K_{W(E)}$, it is easily seen that $K(E)$ is the greatest fundamental orthodox semigroup having E as its band of idempotents. Write $\bar{K}(E)$ for $Q_{K(E)}$, and let $\eta: K(E) \to \bar{K}(E)$ be the natural homomorphism.

If Q is a semilattice of groups, let us call a homomorphism $\psi: Q \to \bar{K}(E)$ *idempotent-bijective* if it induces an isomorphism of the semilattice of idempotents of Q onto that of $\bar{K}(E)$. The following is an immediate consequence of these two results of Hall.

Theorem (T.E. Hall). *Let E be a band and Q a semilattice of groups which are compatible. If $\psi: Q \to \bar{K}(E)$ is an idempotent-bijective homomorphism, then the spined product of Q and $K(E)$ relative to ψ and η is an orthodox extension of E by Q. Conversely, every orthodox extension of E by Q is obtained in this way from some idempotent-bijective homomorphism $\psi: Q \to \bar{K}(E)$.*

We give below (Theorem 8.2) an explicit description of $K(E)$ and $\bar{K}(E)$. Theorem 1.1 would follow immediately from this and Hall's theorem, but we have chosen to give a direct proof thereof (sections 2–4), using the "theorem of vertical associativity" [1; p. 302] instead of Hall's theorem. Hall has an explicit expression for $K(E)$ as a subsemigroup of $W(E)$, and it is possible to derive Theorem 8.2 from this, and hence also Theorem 1.1. Unfortunately, the author's derivation is just as long as the proof given below, and it was thought better to defer publication until a shorter one is found.

In section 2, it is shown that Theorem 1.1 is just a refinement of Preston's theorem, and likewise of Yamada's theorem. In section 5, a variant of Theorem 1.1 is given which similarly refines Fantham's theorem.

In section 7 a reformulation (Theorem 7.3) of Theorem 1.1 in the light of Hall's theorem is given, and the latter is proved in section 8 from Theorem 1.1 (or 7.3).

In the concluding section 9, a characterization is given of those bands E such that every orthogroup S for which $E_S \cong E$ splits, that is, is a spined product of Q_S and E_S. Yamada's theorem [12, 13] that S splits if E_S is normal is an immediate consequence.

Notation and terminology are those of [2] unless otherwise specified.

1. Statement of the main theorem

Recall [2, vol. 1; p. 129] that a band E can be expressed as a semilattice Y of rectangular bands $E_\iota (\iota \in Y)$. Each E_ι is a $\mathcal{D}$-class of E, $\mathcal{J} = \mathcal{D}$, $\mathcal{D}$ is a congruence on E, and $E/\mathcal{D} \cong Y$. We call Y the *structure semilattice* of E. For e in E, we usually write $E(\leqslant e)$ for $eEe = \{f \in E : f \leqslant e\}$.

For e in E, let $T_{e,e}$ be the group of automorphisms of $E(\leqslant e)$, and let

$$K_e = \{\alpha \in T_{e,e} : x\alpha \mathcal{D} x \text{ for all } x \text{ in } E(\leqslant e)\} .$$

Clearly K_e is a subgroup of $T_{e,e}$ and consists of all automorphisms of $E(\leqslant e)$ which leave invariant all the sets $E_\iota \cap E(\leqslant e)$, $(\iota \in Y)$. We call K_e the *$\mathcal{D}$-restricted automorphism group of $E(\leqslant e)$.*

Recall [2; Theorem 4.11] that any inverse semigroup Q which is a union of groups can be represented as a semilattice Y of groups $G_\iota(\iota \in Y)$. All five Green's relations coincide on Q, each G_ι is an $\mathcal{H}$-class (and a $\mathcal{J}$-class), $\mathcal{H}$ is a congruence on Q, and $Q/\mathcal{H} \cong Y$. The set E_Q of idempotents of Q is just the set $\{1_\iota : \iota \in Y\}$, where 1_ι is the identity element of G_ι, and E_Q is a subsemigroup of Q isomorphic to Y. For each pair of elements ι, κ of Y with $\iota \geqslant \kappa$, a homomorphism $\phi_{\iota,\kappa}: G_\iota \to G_\kappa$ is defined by $a\phi_{\iota,\kappa} = a1_\kappa\ (= 1_\kappa a)$, for all a in G_ι. We call $\{\phi_{\iota,\kappa} : \iota \geqslant \kappa\}$ the set of *connecting homomorphisms* of Q; they determine product in Q as follows: if $a \in G_\iota$ and $b \in G_\kappa$ (ι, κ in Y), then

$$ab = (a\phi_{\iota,\iota\kappa})(b\phi_{\kappa,\iota\kappa}) , \tag{1.1}$$

where $\iota\kappa$ is the product (or meet) of ι and κ in Y.

The following is the main theorem of the present paper.

Theorem 1.1. *Let E be a band and Q a semilattice Y of groups $G_\iota(\iota \in Y)$, such that the structure semilattice of E is isomorphic to Y, so that E can be represented as the semilattice Y of rectangular bands $E_\iota(\iota \in Y)$. Let $\{\phi_{\iota,\kappa} : \iota \geqslant \kappa\}$ be the set of connecting homomorphisms for Q.*

For each ι in Y, select a representative u_ι in E_ι, and denote by K_ι the $\mathcal{D}$-restricted automorphism group K_{u_ι} of $E(\leqslant u_\iota)$. For each ι in Y, let $\psi_\iota: G_\iota \to K_\iota$ be a homomorphism, such that if $\iota \geqslant \kappa$ in Y, $a \in G_\iota$, and $x \in E(\leqslant u_\kappa)$, then

$$x(a\phi_{\iota,\kappa}\psi_\kappa) = u_\kappa(u_\iota x u_\iota)(a\psi_\iota)u_\kappa . \tag{1.2}$$

Let $S_\iota = G_\iota \times E_\iota$, and $S = \bigcup\{S_\iota : \iota \in Y\}$. Define product in S as follows. If $(a, e) \in S_\iota$ and $(b, f) \in S_\kappa$ (for any ι, κ in Y), then

$$(a, e)(b, f) = (ab, \bar{e}\bar{f}) , \tag{1.3}$$

where ab is the product of a and b in Q, as given by (1.1), *and $\bar{e}\bar{f}$ is the product of $\bar{e}$ and $\bar{f}$ in $E_{\iota\kappa}$, and where*

$$\bar{e} = e(u_\iota efeu_\iota)(a^{-1}\psi_\iota)e , \tag{1.4a}$$

(1.4b) $\bar{f} = f(u_\kappa fefu_\kappa)(b\psi_\kappa)f$.

Then S is an orthogroup, and every orthogroup can be constructed in this way, taking E to be its band of idempotents and Q its greatest inverse homomorphic image.

It is important to remark that the orthogroup S constructed in Theorem 1.1 depends not only on E, Q, and the set $\{\psi_\iota : \iota \in Y\}$ of homomorphisms, but also upon the manner of indexing the rectangular band components of E by Y. Each indexing amounts to specifying a homorphism of E onto Y. We give an example to show that two different indexings can lead to non-isomorphic orthogroups.

Let $Y = \{\iota, \kappa, \lambda\}$ with $\iota > \lambda$, $\kappa > \lambda$, and $\iota\kappa = \kappa\iota = \lambda$. Let $Q = G_\iota \cup G_\kappa \cup G_\lambda$, where

$$G_\iota = \{1_\iota, a\},\ G_\kappa = \{1_\kappa, b, b^2, b^3\},\ G_\lambda = \{1_\lambda\}$$

are cyclic groups of orders 2,4 and 1, respectively, and where $G_\iota G_\kappa = G_\kappa G_\iota = G_\lambda$, with 1_λ acting as the zero element of Q. Let $E = \{e, f, g, h\}$ with product given by the table. We find that $E(\leqslant e) = \{e, g, h\}$, $E(\leqslant f) = \{f, g\}$, $K_e = \{\epsilon_e, \alpha\}$, $K_f = \{\epsilon_f\}$,

	e	f	g	h
e	e	g	g	h
f	g	f	g	h
g	g	g	g	h
h	h	g	g	h

where ϵ_e denotes the identity element of K_e, and α is the transposition (gh). There are two ways of indexing the rectangular components of E:

(i) $E_\iota = \{e\}$, $E_\kappa = \{f\}$, $E_\lambda = \{g, h\}$;
(ii) $E'_\iota = \{f\}$, $E'_\kappa = \{e\}$, $E'_\lambda = \{g, h\}$.

We indicate the second way by primes. Define the homomorphisms ψ and ψ' for the two cases as follows:

(i) $a\psi_\iota = \alpha$, $b\psi_\kappa = \epsilon_f$;
(ii) $a\psi'_\iota = \epsilon_f$, $b\psi'_\kappa = \alpha$.

(Since K_λ is trivial, (1.2) puts no restriction on the ψ's.) Denote by S and S' the resulting orthogroups. As sets,

$$S = (G_\iota \times \{e\}) \cup (G_\kappa \times \{f\}) \cup (G_\lambda \times \{g, h\}),$$

$$S' = (G_\iota \times \{f\}) \cup (G_\kappa \times \{e\}) \cup (G_\lambda \times \{g, h\}).$$

Each is a union of a cyclic group of order 2, a cyclic group of order 4, and a right zero semigroup of order 2. An isomorphism of S onto S' would have to map the element (b, f) of S onto an element of S' of order 4, hence onto (b, e) or (b^3, e). In S we find that

$$(1_\lambda, g)(b, f) = (1_\lambda, g),\ (1_\lambda, h)(b, f) = (1_\lambda, g)\,,$$

while in S' we find that

$$(1_\lambda, g)(b, e) = (1_\lambda, h),\ (1_\lambda, h)(b, e) = (1_\lambda, g)$$

$$(1_\lambda, g)(b^3, e) = (1_\lambda, h),\ (1_\lambda, h)(b^3, e) = (1_\lambda, g)\,.$$

Thus (b, e) and (b^3, e) induce a permutation of the kernel, while (b, f) does not. Clearly, no such isomorphism can exist.

2. Two-component orthogroups; orthogroupoids

The purpose of this section is three-fold: (1) to examine the important special case of two-component orthogroups; (2) to show that the definition (1.3) of product is not different from that in the Preston Theorem, but only more detailed; (3) to prepare for the proof of the direct part of Theorem 1.1 in the next section by showing that the groupoid S defined in the theorem is an "orthogroupoid" as defined in [1], even without condition (1.2).

With this last purpose in mind, we begin by assuming all the hypotheses of Theorem 1.1 except (1.2).

Lemma 2.1. *If $e \in E$ and $f, g \in E$ with $\iota \geqslant \kappa$, then $feg = fg$.*

Proof. Recall the elementary fact that $xyz = xz$ if x, y, z are elements of a rectangular band. Since f, feg, and g all belong to E_κ, we have $feg = f(feg)g = fg$.

Lemma 2.2. *If $(a, e) \in S_\iota$, $(b, f) \in S_\kappa$, and $\iota \geqslant \kappa$ in Y, then*

(2.1a) $$(a, e)(b, f) = ((a\phi_{\iota,\kappa})b,\ e(u_\iota efeu_\iota)(a^{-1}\psi_\iota)ef)\,,$$

(2.1b) $$(b, f)(a, e) = (b(a\phi_{\iota,\kappa}), fe(u_\iota efeu_\iota)(a\psi_\iota)e)\,.$$

Proof. When $\iota \geqslant \kappa$, (1.4b) gives $\bar{f} = fxf$ with $x = (u_\kappa fefu_\kappa)(b\psi_\kappa)$ in E_κ, hence $\bar{f} = f$. (2.1a) is then immediate from (1.3) and (1.4a). The proof of (2.1b) is similar; since (a, e) is now the right-hand factor, $a\psi_\iota$ appears instead of $a^{-1}\psi_\iota$.

Lemma 2.3. *If $(a, e), (b, f) \in S$, then*

$$(a, e)(b, f) = (ab, ef),$$

and hence S_ι is the direct product of the group G_ι and the rectangular band E_ι.

Proof. When $\iota = \kappa$, (1.4) gives $\bar{e} = e$ and $\bar{f} = f$.

Recall that the direct product of a group and a rectangular band is called a *rectangular group.*

Denote by 1_ι the identity element of the group G_ι. From Lemma 2.3, it is clear that the idempotents of S_ι are the elements $(1_\iota, e)$ with e in E_ι.

Lemma 2.4. *Let $\iota, \kappa \in Y$, and let $e \in E_\iota$ and $f \in E_\kappa$. Then*

$$(1_\iota, e)(1_\kappa, f) = (1_{\iota\kappa}, ef).$$

Hence the set E_S of idempotents of the groupoid S is $\bigcup \{1_\iota \times E_\iota : \iota \in Y\}$, E_S is a subsemigroup of S, and $e \longrightarrow (1_\iota, e)$ if $e \in E_\iota$ is an isomorphism of the band E onto E_S.

Proof. $1_\iota \psi_\iota$ must be the identity map of $E(\leqslant u_\iota)$. Hence (1.4a) in the present case gives, using Lemma 2.1,

$$\bar{e} = e(u_\iota efeu_\iota)e = (eu_\iota e)f(eu_\iota e) = efe.$$

Similarly, $\bar{f} = fef$. By (1.1), $1_\iota 1_\kappa = 1_{\iota\kappa}$. Hence (1.3) gives

$$(1_\iota, e)(1_\kappa, f) = (1_{\iota\kappa}, efefef) = (1_{\iota\kappa}, ef).$$

The remaining assertions are obvious from this.

We shall on occassion identify e with $(1_\iota, e)$ when $e \in E_\iota$, thus E_S with E.

Recall [1, pp. 297–298] that an *orthogroupoid* is a groupoid which is a semilattice Y of groupoids S_ι ($\iota \in Y$) satisfying the following conditions:

(A_1) Each S_ι is a rectangular group.

(A_2) The set E_S of idempotent elements of S is a subsemigroup of S.

(A_3) For every pair ι, κ of elements of Y such that $\iota > \kappa$, the subgroupoid $S_\iota \cup S_\kappa$ of S is associative.

(A_4) If $a \in H_e$ and $b \in H_f$ (e, f in E_S), then

$$ab = (a \cdot ef)(ef \cdot b).$$

(A_1) follows from Lemma 2.3, and (A_2) from Lemma 2.4. Let us show that

(A_4) holds. In our present notation, (A_4) reads: if $(a, e) \in S_\iota$ and $(b, f) \in S_\kappa$, then

$$(2.2) \qquad (a, e)(b, f) = (a, e)(1_{\iota\kappa}, ef) \cdot (1_{\iota\kappa}, ef)(b, f) .$$

By Lemma 2.2,

$$(a, e)(1_{\iota\kappa}, ef) = (a\phi_{\iota,\iota\kappa}, i)$$

and

$$(1_{\iota\kappa}, ef)(b, f) = (b\phi_{\kappa,\iota\kappa}, j)$$

where $\quad i = e(u_\iota e(ef)eu_\iota)(a^{-1}\psi_\iota)e(ef)$

$\qquad = e(u_\iota efeu_\iota)(a^{-1}\psi_\iota)ef$

and $\quad j = (ef)f(u_\kappa f(ef)fu_\kappa)(b\psi_\kappa)f$

$\qquad = ef(u_\kappa fefu_\kappa)(b\psi_\kappa)f .$

Clearly $ij = \bar{e}\bar{f}$, where $\bar{e}$ and $\bar{f}$ are given by (1.4), and hence the right hand side of (2.2) reduces by Lemma 2.3, to $(a\phi_{\iota,\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa}, \bar{e}\bar{f})$, which equals $(a, e)(b, f)$ by (1.3).

To show that S is an orthogroupoid, it remains to show that (A_3) holds. So let $\iota > \kappa$ in Y, and consider the subgroupoid $S_\iota \cup S_\kappa$ of S. Write E_κ as a direct product $E_\kappa = I_\kappa \times \Lambda_\kappa$ of a left zero semigroup I_κ and a right zero semigroup Λ_κ. We use Lemmas 2 and 3 of [1], combining them into the following lemma, with the notation changed to accord with that we are now using.

Lemma 2.5. *Let $S_\iota = G_\iota \times E_\iota$ and $S_\kappa = G_\kappa \times E_\kappa = G_\kappa \times I_\kappa \times \Lambda_\kappa$ be disjoint rectangular groups. Assume that there exist*

(a) *a left representation t of S_ι by transformations of I_κ,*

(b) *a right representation τ of S_ι by transformations of Λ_κ,*
and

(c) *a homomorphism ϕ of G_ι into G_κ.*

Define a binary operation on $S_\iota \cup S_\kappa$ by keeping the given ones on S_ι and S_κ, and defining the products AB and BA of elements $A = (a, e)$ of S_ι and $B = (b; i, \lambda)$ of S_κ as follows:

$$(2.3a) \qquad AB = ((a\phi)b; (tA)i, \lambda) ,$$

$$(2.3b) \qquad BA = (b(a\phi); i, \lambda(A\tau)) .$$

Then $S_\iota \cup S_\kappa$ becomes a (two-component) orthogroup, with S_κ an ideal. Conversely, *every possible associative binary operation on $S_\iota \cup S_\kappa$ extending the given ones on S_ι and S_κ, and such that S_κ is an ideal, can be constructed in this way.*

We may write Aj instead of $(tA)i$, and λA instead of $\lambda(A\tau)$; (a) and (b) are then equivalent to requiring that

$$(AA')i = A(A'i),\ \lambda(AA') = (\lambda A)A',$$

for all A, A' in S_ι, i in I_κ, and λ in Λ_κ.

Let us identify E_ι with $\{1_\iota\} \times E_\iota$ and E_κ with $\{1_\kappa\} \times E_\kappa$. Then $E_\iota \cup E_\kappa$ is a (given) band, and, by the converse part of Lemma 2.5, there exist a left action of E_ι on I_κ and a right action of E_ι on Λ_κ,

$$(ee')i = e(e'i),\ \lambda(ee') = (\lambda e)e',$$

(e, e' in E_ι, i in I_κ, λ in Λ_κ), such that

(2.4a) $\quad e(i, \lambda) = (ei, \lambda)$,

(2.4b) $\quad (i, \lambda)e = (i, \lambda e)$,

for all e in E_ι and all (i, λ) in $I_\kappa \times \Lambda_\kappa$.

Products in $S_\iota S_\kappa$ and $S_\kappa S_\iota$ are given by Lemma 2.2 above. Let us translate equations (2.1) into the present notation. By (2.4),

$$u_\iota(i, \lambda)u_\iota = (u_\iota i, \lambda u_\iota),$$

so that $u_\iota E_\kappa u_\iota = u_\iota I_\kappa \times \Lambda_\kappa u_\iota$. If α is an automorphism of $u_\iota E_\kappa u_\iota$, then there exist permutations α' of $u_\iota I_\kappa$ and α'' of $\Lambda_\kappa u_\iota$ such that

(2.5) $\quad (i, \lambda)\alpha = (i\alpha', \lambda\alpha'')$, for all (i,λ) in $u_\iota I_\kappa \times \Lambda_\kappa u_\iota$.

Conversely, if α' is any permutation of $u_\iota I_\iota$, and α'' is any permutation of $\Lambda_\kappa u_\iota$, then (2.5) defines an automorphism of $u_\iota E_\kappa u_\iota$.

Let $a \in G_\iota$. Since $a\psi_\iota$ induces an automorphism of $u_\iota E_\kappa u_\iota$, there exist permutations $a\psi'_{\iota,\kappa}$ and $a\psi''_{\iota,\kappa}$ of $u_\iota I_\kappa$ and $\Lambda_\kappa u_\iota$, respectively, such that

(2.6) $\quad (i,\lambda)(a\psi_\iota) = (i(a\psi'_{\iota,\kappa}), \lambda(a\psi''_{\iota,\kappa}))$, for all (i,λ) in $u_\iota I_\kappa \times \Lambda_\kappa u_\iota$.

Since ψ_ι is a homomorphism, $\psi'_{\iota,\kappa}: G_\iota \to \mathcal{G}(u_\iota I_\kappa)$ and $\psi''_{\iota,\kappa}: G_\iota \to \mathcal{G}(\Lambda_\kappa u_\iota)$ are likewise homomorphisms, where $\mathcal{G}(X)$ denotes the symmetric group on X.

Let $(a, e) \in S$ and $(b, f) \in S$, and let $f = (i, \lambda)$. By (2.4) and (2.6),

$$(u_\iota efeu_\iota)(a^{-1}\psi_\iota) = ((u_\iota ei)(a^{-1}\psi'_{\iota,\kappa}), (\lambda eu_\iota)(a^{-1}\psi''_{\iota,\kappa})).$$

Hence

$$e(u_\iota efeu_\iota)(a^{-1}\psi_\iota)ef = (e(u_\iota ei)(a^{-1}\psi'_{\iota,\kappa}), (\lambda eu_\iota)(a^{-1}\psi''_{\iota,\kappa})e)(i,\lambda)$$
$$= (e(u_\iota ei)(a^{-1}\psi'_{\iota,\kappa}), \lambda),$$

and so, by (2.1a)

$$(a, e)(b; i, \lambda) = (a\phi_{\iota,\kappa} \cdot b; e(u_\iota ei)(a^{-1}\psi'_{\iota,\kappa}), \lambda).$$

Comparing this with (2.3a), we can take ϕ to be $\phi_{\iota,\kappa}$, so condition (c) of Lemma 2.5 holds, and we can define t by

$$\text{(2.7a)} \qquad [t(a, e)]\, i = e(u_\iota ei)(a^{-1}\psi'_{\iota,\kappa}).$$

Similarly, on calculating (2.1b), it becomes the same as (2.3b) if we define τ by

$$\text{(2.7b)} \qquad \lambda[(a, e)\tau] = (\lambda eu_\iota)(a\psi''_{\iota,\kappa})e.$$

To show that t and τ so defined satisfy conditions (a) and (b) of Lemma 2.5, let (a, e) and (a', e') be elements of S_ι and let $i \in I_\kappa$.

First note that

$$u_\iota ee'u_\iota = u_\iota, \quad eu_\iota = ee'u_\iota, \quad u_\iota e' = u_\iota ee',$$

by Lemma 2.1, so that if $j \in u_\iota I_\kappa$,

$$u_\iota ee'j = u_\iota ee'u_\iota j = u_\iota j = j,$$

$$ej = eu_\iota j = ee'u_\iota j = ee'j.$$

Then, by (2.7a),

$$[t(a,e) \cdot t(a', e')]\, i = t(a, e)\,[e'(u_\iota e'i)(a^{-1}\psi'_{\iota,\kappa})]$$
$$= e[u_\iota ee'(u_\iota e'i)(a'^{-1}\psi'_{\iota,\kappa})]\,(a^{-1}\psi'_{\iota,\kappa})$$
$$= e \cdot (u_\iota e'i)(a'^{-1}\psi'_{\iota,\kappa})(a^{-1}\psi'_{\iota,\kappa})$$
$$= ee' \cdot (u_\iota ee'i)\,((aa')^{-1}\psi'_{\iota,\kappa})$$
$$= t(aa', ee')i.$$

Similarly, we can show that τ satisfies (b). The associativity of $S_\iota \cup S_\kappa$ now follows from Lemma 2.5 Hence (A_3) holds for S, and we have proved the following.

Proposition 2.6. *Assuming all the hypotheses of Theorem* 1.1 *except* (1.2), *the groupoid S there constructed is an orthogroupoid.*

The definition of product in Preston's Theorem [1; eq. (6), p. 289] is obtained from (2.3) by using (2.2), while that in Theorem 1.1 can similarly be obtained from (2.1) and (2.2). Since (2.1) is actually the same as (2.3), with t and τ given by (2.7), we see that the definition of product in Theorem 1.1 differs from that in Preston's Theorem only in that representations t and τ are now expressed by (2.7) in terms of the ψ_ι's.

If E is a two-component band, $E = E_\iota \cup E_\kappa$, with $Y = \{\iota, \kappa\}$, where $\iota > \kappa$, then in the foregoing notation, K_ι consists of *all* automorphisms of $u_\iota E_\kappa u_\iota$, so that $K_\iota \cong \mathcal{G}(u_\iota I_\kappa) \times \mathcal{G}(\Lambda_\kappa u_\iota)$. We also see at once that K_κ is trivial. If $Q = G_\iota \cup G_\kappa$ is a given two-component semilattice of groups, then the only variable in the construction of all possible orthodox extensions S of E by Q is the homomorphism $\psi_\iota \colon G_\iota \to K_\iota$. This in turn decomposes into the entirely arbitrary homomorphisms $\psi'_{\iota,\kappa}\colon G_\iota \to \mathcal{G}(u_\iota I_\kappa)$ and $\psi''_{\iota,\kappa}\colon G_\iota \to \mathcal{G}(\Lambda_\kappa u_\iota)$. Thus the structure of S depends upon two representations of G_ι by permutations (of $u_\iota I_\kappa$ and $\Lambda_\kappa u_\iota$), which are entirely arbitrary and independent of each other.

In the foregoing, we have proved the direct part of the following proposition; the converse part will follow immediately once we have shown the converse part of Theorem 1.1.

Proposition 2.7. *Let $S_\iota = G_\iota \times E_\iota$ and $S_\kappa = G_\kappa \times E_\kappa = G_\kappa \times I_\kappa \times \Lambda_\kappa$ be two disjoint rectangular groups. Let $\phi_{\iota,\kappa}\colon G_\iota \to G_\kappa$ be a homomorphism of groups. Let $(e,i) \to ei$ be a left action of E_ι on I_κ; that is, $e(e'i) = (ee')i$ for all e, e' in E_ι and i in I_κ. Let $(\lambda, e) \to \lambda e$ be a right action of E_ι on Λ_κ; that is, $(\lambda e)e' = \lambda(ee')$ for all e, e' in E_ι and λ in Λ_κ.*

Choose and fix an element u_ι of E_ι. Let $\psi'_{\iota,\kappa}\colon G_\iota \to \mathcal{G}(u_\iota I_\kappa)$ and $\psi''_{\iota,\kappa}\colon G_\iota \to \mathcal{G}(\Lambda_\kappa u_\iota)$ be representations of G_ι by permutations of the sets $u_\iota I_\kappa$ and $\Lambda_\kappa u_\iota$, respectively.

Let $S = S_\iota \cup S_\kappa$. Define product on S by retaining those in the rectangular groups S_ι and S_κ, and defining the products AB and BA of elements $A = (a, e)$ of S_ι and $B = (b; i, \lambda)$ of S_κ as follows:

$$AB = ((a\phi_{\iota,\kappa})b;\, e(u_\iota ei)(a^{-1}\psi'_{\iota,\kappa}),\, \lambda)\,,$$

$$BA = (b(a\phi_{\iota,\kappa});\, i,\, (\lambda e u_\iota)(a\psi''_{\iota,\kappa})e)\,.$$

Then S becomes a (two-component) orthogroup, with S_κ an ideal. Conversely, every possible associative binary operation on S extending the given ones on S_ι and S_κ, and such that S_κ is an ideal, can be constructed in this way.

We remark in conclusion that Theorem 1.1 also refines Yamada's theorem [14].

Writing (a, e) instead of (e, ξ) to accord with present notation, his definition of product [14; p. 5] is

$$(a, e)(b, f) = (ab, \delta_{a,b}(e, f)).$$

Comparing with (1.3), we see that $\delta_{a,b}(e, f) = \bar{e}\bar{f}$, with $\bar{e}$ and $\bar{f}$ given by (1.4).

3. Proof of the direct part of Theorem 1.1

Having shown that S is an orthogroupoid (Proposition 2.6), we can use Theorem VA [1, p. 302], the "theorem of vertical associativity", to show that S is an orthogroup, and thus complete the proof of the direct part of Theorem 1.1. There are four conditions that must be proved; two of these, in present notation, are the following:

(17-ABC) $\quad (a, e) \cdot (1_\kappa, f)(1_\lambda, g) = (a, e)(1_\kappa, f) \cdot (1_\lambda, g)$,

(17-BAC) $\quad (1_\kappa, f) \cdot (a, e)(1_\lambda, g) = (1_\kappa, f)(a, e) \cdot (1_\lambda, g)$,

where $\iota > \kappa > \lambda$ in Y, $a \in G_\iota$, $e \in E_\iota$, $f \in E_\kappa$, $g \in E_\lambda$.

(17-CBA) is left–right dual to (17-ABC), and (17-CAB) to (17-BAC). Since our definition, (1.3) and (1.4), of product in S is left–right symmetric, it suffices to prove (17-ABC) and (17-BAC).

Lemma 3.1. *Let $\iota \geqslant \kappa \geqslant \lambda$ in Y; $\alpha \in K_\iota$; $f, f' \in E_\kappa$; $g, g' \in E_\lambda$. Then*

(3.1) $$(u_\iota fgf'u_\iota)\alpha g' = (u_\iota fgu_\iota)\alpha g'.$$

Proof. Since $\iota \geqslant \lambda$, we can write $g = gu_\iota g = gu_\iota \cdot u_\iota g$. Hence

$$\begin{aligned}(u_\iota fgf'u_\iota)\alpha g' &= (u_\iota fgu_\iota \cdot u_\iota gf'u_\iota)\alpha g' \\ &= (u_\iota fgu_\iota)\,\alpha(u_\iota gf'u_\iota)\alpha g'\end{aligned}$$

which reduces to the right-hand side of (3.1) by Lemma 2.1.

In the proofs that follow, the symbol $\sqcup$ under part of an expression indicates that this part can be omitted because of Lemma 2.1.

Proof of (17-ABC). We have

$$(a, e) \cdot (1_\kappa, f)(1_\lambda, g) = (a, e)(1_\lambda, fg) = (a\phi_{\iota,\lambda}, i),$$

where by (2.1a),

$$i = e(u_\iota efgeu_\iota)(a^{-1}\psi_\iota)efg\,.$$

By Lemma 3.1,

$$i = e(u_\iota efgu_\iota)(a^{-1}\psi_\iota)g\,.$$

Likewise,

$$(a, e)(1_\kappa, f)\cdot(1_\lambda, g) = (a\phi_{\iota,\kappa}, \bar{e}f)(1_\lambda, g) = (a\phi_{\iota,\kappa}\phi_{\kappa,\lambda}, j)\ ,$$

where, by (2.1a),

$$\bar{e} = e(u_\iota efeu_\iota)(a^{-1}\psi_\iota)e$$

and

$$j = \bar{e}f(u_\kappa \bar{e}fg\bar{e}fu_\kappa)(a^{-1}\phi_{\iota,\kappa}\psi_\kappa)\bar{e}fg\ .$$

Since $\phi_{\iota,\kappa}\phi_{\kappa,\lambda} = \phi_{\iota,\lambda}$ by one of the hypotheses of Theorem 1.1, all that remains is to show that $i = j$. Using first (1.2), then Lemma 2.1, then Lemma 3.1 (to remove $\bar{e}fu_\kappa$),

$$\begin{aligned} j &= \bar{e}fu_\kappa(u_\iota u_\kappa \bar{e}fg\bar{e}fu_\kappa u_\iota)(a^{-1}\psi_\iota)u_\kappa\bar{e}fg \\ &= \bar{e}u_\kappa(u_\iota u_\kappa fgu_\iota)(a^{-1}\psi_\iota)g \\ &= e(u_\iota efeu_\iota)(a^{-1}\psi_\iota)eu_\kappa(u_\iota u_\kappa fgu_\iota)(a^{-1}\psi_\iota)g\ . \end{aligned}$$

We can replace eu_κ by $u_\iota eu_\kappa u_\iota$ since its neighbors are in $E(\leqslant u_\iota)$. Then, using the fact that $a^{-1}\psi_\iota$ is an automorphism of $E(\leqslant u_\iota)$ with inverse $a\psi_\iota$,

$$\begin{aligned} j &= e[(u_\iota efeu_\iota)\cdot(u_\iota eu_\kappa u_\iota)(a\psi_\iota)\cdot(u_\iota u_\kappa fgu_\iota)]\,(a^{-1}\psi_\iota)g \\ &= e(u_\iota efgu_\iota)(a^{-1}\psi_\iota)g = i\,. \end{aligned}$$

Proof of (17-BAC). We have

$$(1_\kappa, f)\cdot(a, e)(1_\lambda, g) = (1_\kappa, f)(a\phi_{\iota,\lambda}, h) = (a\phi_{\iota,\lambda}, i)\,,$$

where, by (2.1),

$$h = e(u_\iota egeu_\iota)(a^{-1}\psi_\iota)eg$$

and

$$\begin{aligned} i &= f(u_\kappa fhfu_\kappa)(1_\kappa\psi_\kappa)fh \\ &= \underbrace{fu_\kappa} fh\underbrace{fu_\kappa} fh = fhfh = fh \\ &= fe(u_\iota egeu_\iota)(a^{-1}\psi_\iota)g \,. \end{aligned}$$

By Lemma 3.1,

$$i = fe(u_\iota egu_\iota)(a^{-1}\psi_\iota)g \;.$$

Also, $$\begin{aligned} (1_\kappa, f)(a, e)\cdot(1_\lambda, g) &= (a\phi_{\iota,\kappa}, f\bar{e})(1_\lambda, g) \\ &= (a\phi_{\iota,\kappa}\phi_{\kappa,\lambda}, j) \;, \end{aligned}$$

where, by (2.1),

$$\bar{e} = e(u_\iota efeu_\iota)(a\psi_\iota)e$$

and $$j = f\bar{e}(u_\kappa f\bar{e}gf\bar{e}u_\kappa)(a^{-1}\phi_{\iota,\kappa}\psi_\kappa)f\bar{e}g \;.$$

By (2.1), and then using Lemma 3.1 (to remove $f\bar{e}u_\kappa$),

$$\begin{aligned} j &= \underbrace{f\bar{e}}u_\kappa(u_\iota \underbrace{u_\kappa f}\bar{e}gf\bar{e}u_\kappa u_\iota)(a^{-1}\psi_\iota)\underbrace{u_\kappa f\bar{e}}g \\ &= fu_\kappa(u_\iota u_\kappa\bar{e}gu_\iota)(a^{-1}\psi_\iota)g \\ &= fu_\kappa\,[u_\iota u_\kappa e(u_\iota efeu_\iota)(a\psi_\iota)egu_\iota]\,(a^{-1}\psi_\iota)g \\ &= \underbrace{fu_\kappa(u_\iota u_\kappa eu_\iota)(a^{-1}\psi_\iota)\cdot(u_\iota efeu_\iota)}\cdot(u_\iota egu_\iota)(a^{-1}\psi_\iota)g \\ &= feu_\iota\cdot(u_\iota egu_\iota)(a^{-1}\psi_\iota)g = i\,. \end{aligned}$$

Since $a\phi_{\iota,\lambda} = a\phi_{\iota,\kappa}\phi_{\kappa,\lambda}$, this concludes the proof of (17-BAC), and hence that of the direct part of Theorem 1.1.

4. Proof of the converse part of Theorem 1.1

Let S be an orthogroup and E_S its band of idempotents. Then [1,9,10,13,14] S is a semilattice Y of rectangular groups $S_\iota(\iota \in Y)$. We know that $E_\iota = E_S \cap S_\iota$ is a rectangular band.

For a, b in S, define $a \succ b$ (or $b \prec a$) to mean that $a \in S_\iota$, $b \in S_\kappa$, with $\iota \geqslant \kappa$; and define $a \sim b$ to mean that a and b belong to the same component S_ι of S. For e, f in E_S, $e \succ f$ if and only if $fef = f$.

Lemma 4.1. *If $a,b \in S$, $e \in E_S$, and $e \succ a \sim b$, then $aeb = ab$.*

Proof. $a \in H_f$ and $b \in H_g$ for some f, g in E_S. Since f, feg, g all belong to the same rectangular band, $feg = f(feg)g = fg$. Hence $aeb = afegb = afgb = ab$.

For $e \succ f$ in E_S define $\phi_{e,f}: H_e \to H_f$ by

$$(4.1) \qquad a\phi_{e,f} = faf \quad (\text{all } a \text{ in } H_e).$$

$\phi_{e,f}$ is the same as Fantham's $\eta_{f,e}$ [1, p. 309]. The following is due to Fantham [3].

Lemma 4.2. *(i) $\phi_{e,f}$ is a homorphism. (ii) $\phi_{e,e}$ is the identity map of H_e. (iii) If $e \succ f \succ g$ in E_S, then*

$$\phi_{e,f}\phi_{f,g} = \phi_{e,g}.$$

Proof. To show (i), let $a, b \in H_e$. By Lemma 4.1, since $e \succ f$,

$$(a\phi_{e,f})(b\phi_{e,f}) = faf \cdot fbf = (fa)f(bf) = fabf = (ab)\phi_{e,f}.$$

(ii) is evident. To show (iii), let $a \in H_e$. Using Lemma 4.1 twice,

$$a\phi_{e,f}\phi_{f,g} = gfafg = gafg = gag = a\phi_{e,g}.$$

For a in S, denote by a^{-1} the inverse of a in the group H_a. For each e in E_S and each a in H_e, let $a\psi_e$ be the transformation of $E_S\,(\leqslant e)$ defined by

$$(4.2) \qquad x(a\psi_e) = a^{-1}xa, \quad (\text{all } x \text{ in } E_S\,(\leqslant e)).$$

In the notation of Hall [5], $a\psi_e = \theta_{a^{-1},a}$. As in section 1, let K_e be the group of $\mathcal{D}$-restricted automorphisms of $E\,(\leqslant e)$.

Lemma 4.3. *For each e in E_S, the mapping $a \to a\psi_e$ ($a \in H_e$) is a homomorphism of H_e into K_e.*

Proof. If $x, y \in E_S\,(\leqslant e)$,

$$(a^{-1}xa)(a^{-1}ya) = a^{-1}xeya = a^{-1}xya,$$

so $a\psi_e$ is a homomorphism of $E_S\,(\leqslant e)$ into itself. It is easily seen that $a^{-1}\psi_e$ is inverse to $a\psi_e$, so $a\psi_e$ is an automorphism of $E_S\,(\leqslant e)$. We proceed to show that it is $\mathcal{D}$-restricted.

Let $x \in E_S(\leqslant e)$. From $aa^{-1}x = ex = x = xe = xaa^{-1}$, we see that $a^{-1}x\,\mathcal{L}_S\,x$ and $xa\,\mathcal{R}_S\,x$, so

$$a^{-1}xa = (a^{-1}x)(xa) \in R_{a^{-1}x} \cap L_{xa}.$$

Hence $a^{-1}xa\mathcal{D}_S x$. But, fo' an orthogroup S, the restriction of $\mathcal{D}_S$ to E_S is just $\mathcal{D}_E$ (which we are denoting by plain $\mathcal{D}$). Hence $x(a\psi_e)\mathcal{D}x$, so $a\psi_e \in K_e$.

If $a,b \in H_e$, then, for all x in $E(\leqslant e)$,

$$x((ab)\psi_e) = b^{-1}a^{-1}xab = x(a\psi_e)(b\psi_e)\,.$$

Hence ψ_e is a homorphism of H_e into K_e.

Lemma 4.4. *Let $e \succ f$ in E_S, and let $a \in H_e$. Then*

(4.3a) $\quad af = (efe)(a^{-1}\psi_e) \cdot a\phi_{e,f}\,,$

(4.3b) $\quad fa = a\phi_{e,f} \cdot (efe)(a\psi_e)\,.$

Proof. By (4.1) and (4.2), and Lemma 4.1,

$$\begin{aligned}(efe)(a^{-1}\psi_e) \cdot a\phi_{e,f} &= a(efe)a^{-1} \cdot faf\\ &= afa^{-1} \cdot af = afef = af\,.\end{aligned}$$

The proof of (4.3b) is similar.

Lemma 4.5. *Let a, $b \in S$. Then $a \in H_e$ and $b \in H_f$, for unique e, f in E_S, and*

(4.4) $\quad ab = (efe)(a^{-1}\psi_e) \cdot a\phi_{e,ef}b\phi_{f,ef} \cdot (fef)(b\psi_f)\,.$

Proof. $ab = aefb = aef \cdot efb$, and (4.4) follows from (4.3).

Lemma 4.6. *Let $e \succ f$ in E_S, $a \in H_e$, and $x \in E(\leqslant f)$. Then*

(4.5) $\quad x(a\phi_{e,f}\psi_f) = f(exe)(a\psi_e)f\,.$

Proof. By (4.1) and (4.2),

$$\begin{aligned}x(a\phi_{e,f}\psi_f) &= (a^{-1}\phi_{e,f})x(a\phi_{e,f})\\ &= fa^{-1}fxfaf = fa^{-1}xaf = fa^{-1}exeaf\\ &= f(exe)(a\psi_e)f\,.\end{aligned}$$

For each ι in Y, choose a representative element u_ι in E_ι, and let $G_\iota = H_{u_\iota}$. Define $\theta: G_\iota \times E_\iota \to S_\iota$ and $\theta': S_\iota \to G_\iota \times E_\iota$ by

(4.6) $\quad (a, e)\theta = eae \quad$ (all (a, e) in $G_\iota \times E_\iota$),

$\quad s\theta' = (u_\iota s u_\iota, e) \quad$ (all s in S_ι), where $e \in E_\iota$ is given by $s \in H_e$. Since $eae \in H_e$ and $u_\iota eaeu_\iota = u_\iota au_\iota = a$, and since $s \in H_e$ implies $eu_\iota su_\iota e = ese = s$, we see that θ and θ' are mutually inverse. Hence θ is a bijection of $G_\iota \times E_\iota$ onto S_ι. Let $(a, e), (b, f) \in G_\iota \times E_\iota$. By Lemma 4.1,

$$eae \cdot fbf = eabf = efabef,$$

that is, $$(a, e)\theta \cdot (b, f)\theta = (ab, ef)\theta = [(a, e)(b, f)]\,\theta \ .$$

Hence θ is an isomorphism of $G_\iota \times E_\iota$ onto S_ι, showing that each S_ι is a rectangular group. We shall identify the "point" eae with its "coordinates" (a, e).

Write $\phi_{\iota,\kappa}$ for $\phi_{u_\iota u_\kappa}$, ψ_ι for ψ_{u_ι}, and K_ι for K_{u_ι}. Then $Q = \bigcup \{G_\iota \colon \iota \in Y\}$ becomes a semilattice of groups with connecting homorphisms $\phi_{\iota,\kappa}$ (which have the requisite properties by Lemma 4.2), when product is defined in Q by (1.1). Condition (1.2) of Theorem 1.1 is immediate from (4.5). All that remains is to show that product in S, as given by (4.4), coincides with (1.3) and (1.4) when we express it in terms of pairs (a, e) and (b, f).

Let $(a, e) \in S_\iota$ and $x \in E(\leqslant e)$. Then

$$\text{(4.7)} \qquad x[(a, e)\,\psi_e] = e(u_\iota xu_\iota)(a\psi_\iota)e \ .$$

For the left-hand side is, by (4.2) and the identification of (a, e) with eae,

$$(eae)^{-1}x(eae) = ea^{-1}exeae = ea^{-1}xae\,,$$

while the right-hand side is, by Lemma 4.1,

$$ea^{-1}(u_\iota xu_\iota)ae = ea^{-1}xae \ .$$

Let $\iota \geqslant \kappa$ in Y, $(a, e) \in S_\iota$, and $f \in E_\kappa$. Then

$$\text{(4.8)} \qquad (a, e)\phi_{e,f} = (a\phi_{\iota,\kappa}, f) \ .$$

For the right-hand side is identified with

$$f(a\phi_{\iota,\kappa})f = fu_\kappa au_\kappa f = faf\,,$$

and the left-hand side with

$$(eae)\phi_{e,f} = feaef = faf.$$

We are now ready to calculate (4.4) in terms of the $\phi_{\iota,\kappa}$ and ψ_ι. Let $(a, e) \in S_\iota$ and $(b, f) \in S_\kappa$ (ι, κ in Y). By(4.7),

$$\text{(4.9a)} \qquad (efe)\,[(a^{-1}, e)\,\psi_e] = e(u_\iota efeu_\iota)(a^{-1}\psi_\iota)e = \bar{e}$$

(4.9b) $\qquad (fef)\,[(b,f)\psi_f] = f(u_\kappa fefu_\kappa)(b\psi_\kappa)f = \bar{f}\,,$

with $\bar{e}$ and $\bar{f}$ the same as given by (1.4). By (4.8), since $ef \in E_{\iota\kappa}$,

$$(a,e)\,\phi_{e,ef} = (a\phi_{\iota,\iota\kappa}, ef)\ ,$$

$$(b,f)\phi_{f,ef} = (b\phi_{\kappa,\iota\kappa}, ef)\ .$$

Since we identify $\bar{e}$ with $(1_{\iota\kappa}, \bar{e})$ and $\bar{f}$ with $(1_{\iota\kappa}, \bar{f})$, (4.4) now gives

$$(a,e)(b,f) = (1_{\iota\kappa}, \bar{e})(a\phi_{\iota,\iota\kappa}, ef)(b\phi_{\kappa,\iota\kappa}, ef)(1_{\iota\kappa}, \bar{f})$$

$$= (a\phi_{\iota,\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa}, \bar{e}ef\bar{f}) = (ab, \bar{e}\bar{f})$$

with ab given by (1.1). Thus (1.3) holds, which concludes the proof of the converse part of Theorem 1.1.

5. A variant of Theorem 1.1 that refines Fantham's Theorem

Theorem 1.1 resembles Preston's Theorem in that it uses only one group G_ι for each component S_ι of S, with correspondingly fewer homomorphisms to consider. In this section we give a variant of Theorem 1.1 which resembles that of Fantham: all the maximal subgroups $H_e (e \in E_S)$ are treated alike, and all the corresponding homomorphisms are taken into consideration. In Theorem 1.1 we keep Preston's $\phi_{\iota,\kappa}$, but express his $t_{\iota,\kappa}$ and $\tau_{\iota,\kappa}$ in terms of the ψ_ι. Here we keep Fantham's $\eta_{f,e}$ (here being denoted by $\phi_{e,f}$), but express his star operation $(*)$ in terms of the ψ_e (eq. (5.3) below). The definition (5.2) of product is precisely that of Fantham [1; eq. (27), p. 310].

If E is a band, and $e, f \in E$, we write $e \succ f$ if $fef = f$, as in section 4. Recall (section 1) that K_e denotes the group of $\mathcal{D}$-restricted automorphisms of $E(\leqslant e)$.

Theorem 5.1. *Let E be a band. For each e in E, let H_e be a group, and assume that $H_e \cap H_f = \emptyset$ if $e \neq f$. For every pair e, f of elements of E such that $e \succ f$, let $\phi_{e,f}$: $H_e \to H_f$ be a homorphism such that $\phi_{e,e}$ is the identity map of H_e, and such that $e \succ f \succ g$ implies $\phi_{e,f}\phi_{f,g} = \phi_{e,g}$.*

Moreover, let ψ_e: $H_e \to K_e$ be a homomorphism, for each e in E, such that if $e \succ f$, $a \in H_e$, and $x \in E(\leqslant f)$, then

(5.1) $\qquad x(a\phi_{e,f}\psi_f) = f(exe)(a\psi_e)f\,.$

Define multiplication in $S = \bigcup\{H_e\colon e \in E\}$ as follows. If $a \in H_e$ and $b \in H_f$, then

(5.2) $\qquad ab = a\phi_{e,a*b} \cdot b\phi_{f,a*b}\ ,$

where product on the right is that in H_{a*b}, *and*

(5.3) $$a*b = (efe)(a^{-1}\psi_e)\cdot(fef)(b\psi_f)\,.$$

Then S is an orthogroup (with $E_S \cong E$*), and conversely, every orthogroup can be constructed in this way.*

Proof. All of the converse part of the theorem has been shown in section 4 (by Lemmas 4.2, 4.3 and 4.6), except for the form (5.2) and (5.3) of product in S. But this follows easily from Lemma 4.5. Letting $g = (efe)(a^{-1}\psi_e)$ and $h = (fef)(b\psi_f)$, and using Lemma 4.1, (4.4) gives

$$\begin{aligned} ab &= g(a\phi_{e,ef})(b\phi_{f,ef})h \\ &= gh(a\phi_{e,ef})gh\cdot gh(b\phi_{f,ef})gh\ . \end{aligned}$$

By (4.1) and Lemma 4.2,

$$\begin{aligned} ab &= (a\phi_{e,ef}\phi_{ef,gh})(b\phi_{f,ef}\phi_{ef,gh}) \\ &= a\phi_{e,gh}\cdot b\phi_{f,gh}\ . \end{aligned}$$

Since, by (5.3), $gh = a*b$, this is just (5.2).

Turning to the direct part, assume all the hypotheses of the theorem. Let $E = \bigcup\{E_\iota\colon \iota\in Y\}$ be the decomposition of E into a semilattice Y of rectangular bands E_ι. For each ι in Y, let $S_\iota = \bigcup\{H_e\colon e\in E_\iota\}$. Let $a, b\in S_\iota$. Then $a\in H_e$ and $b\in H_f$ for some e, f in E_ι. By (5.3), $a*b = e(a^{-1}\psi_e)\cdot f(b\psi_f) = ef$, so, by (5.2)

(5.4) $$ab = a\phi_{e,ef}\cdot b\phi_{f,ef} \text{ when } a\in H_e,\ b\in H_f, \text{ and } e\sim f\,.$$

We proceed to show that each S_ι is a rectangular group. Let u be a fixed element of E_ι. Define $\theta\colon H_u\times E_\iota\to S_\iota$ by $(a, e)\theta = a\phi_{u,e}$. Using (5.4),

$$\begin{aligned} (a,e)\theta\cdot(b,f)\theta &= a\phi_{u,e}\cdot b\phi_{u,f} \\ &= a\phi_{u,e}\phi_{e,ef}\cdot b\phi_{u,f}\phi_{f,ef} \\ &= a\phi_{u,ef}\cdot b\phi_{u,ef} \\ &= (ab)\phi_{u,ef} = (ab, ef)\theta\ . \end{aligned}$$

Hence θ is a homomorphism. If $e\sim f$, then $\phi_{e,f}\phi_{f,e} = \phi_{e,e}$ = identity on H_e, so $\phi_{e,f}$ is an isomorphism. From this we see that θ is injective. If $b\in S_\iota$, then $b\in H_e$ for some e in E_ι. Let $a = b\phi_{e,u}$. Then $(a, e)\theta = b\phi_{e,u}\phi_{u,e} = b$, showing that θ is surjective.

Now select and fix an element u_ι in each E_ι ($\iota\in Y$). Let $G_\iota = H_{u_\iota}$. From the foregoing, the mapping $\theta_\iota: G_\iota \times E_\iota \to S_\iota$ defined by $(a, e)\theta_\iota = a\phi_{u_\iota, e}$ is an isomorphism, and we shall identify the element $a\phi_{u_\iota, e}$ of H_e with its coordinates (a, e).

Write $\phi_{\iota,\kappa}$ for ϕ_{u_ι, u_κ} ($\iota \geqslant \kappa$ in Y). Define product in $Q = \bigcup \{G_\iota: \iota \in Y\}$ by

$$ab = a\phi_{\iota,\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa} \quad \text{if } a \in G_\iota,\ b \in G_\kappa \ . \tag{1.1}$$

Then Q is the semilattice Y of groups G_ι ($\iota \in Y$) with connecting homomorphisms $\phi_{\iota,\kappa}$. For, by hypothesis, $\phi_{\iota,\kappa}: G_\iota \to G_\kappa$ is a homomorphism, $\phi_{\iota,\iota}$ is the identity map of G_ι, and $\iota \geqslant \kappa \geqslant \lambda$ implies $\phi_{\iota,\kappa}\phi_{\kappa,\lambda} = \phi_{\iota,\lambda}$.

Write ψ_ι for ψ_{u_ι} and K_ι for K_{u_ι}. By hypothesis, $\psi_\iota: G_\iota \to K_\iota$ is a homomorphism, and if $\iota \geqslant \kappa$ in Y, $a \in G_\iota$, and $x \in E(\leqslant u_\kappa)$, then by (5.1)

$$x(a\phi_{\iota,\kappa}\psi_\kappa) = u_\kappa(u_\iota x u_\iota)(a\psi_\iota)u_\kappa \ . \tag{1.2}$$

Hence the hypotheses of Theorem 1.1 all hold, and all that remains is to show (1.3) and (1.4). Let $(a, e) \in S_\iota$ and $(b, f) \in S_\kappa$. Since we are identifying these with $a\phi_{u_\iota, e}$ and $b\phi_{u_\kappa, f}$, respectively, (5.3) gives

$$(a, e) * (b, f) = gh$$

where
$$g = (efe)\,[(a^{-1}, e)\,\psi_e] = (efe)(a^{-1}\phi_{u_\iota, e}\psi_e)\,,$$
$$h = (fef)\,[(b, f)\,\psi_f] = (fef)(b\phi_{u_\kappa, f}\psi_f)\ .$$

Using (5.1), we have

$$g = e(u_\iota efeu_\iota)(a^{-1}\psi_\iota)e = \bar{e}\ ,$$
$$h = f(u_\kappa fefu_\kappa)(b\psi_\kappa)f = \bar{f}\ ,$$

with the same $\bar{e}, \bar{f}$ as in (1.4). (5.2) now gives

$$\begin{aligned}
(a, e)(b, f) &= (a, e)\phi_{e,\bar{e}\bar{f}} \cdot (b, f)\phi_{f,\bar{e}\bar{f}} \\
&= a\phi_{u_\iota, e}\phi_{e,\bar{e}\bar{f}} \cdot b\phi_{u_\kappa, f}\phi_{f,\bar{e}\bar{f}} \\
&= a\phi_{u_\iota, \bar{e}\bar{f}} \cdot b\phi_{u_\kappa, \bar{e}\bar{f}} \\
&= a\phi_{u_\iota, u_{\iota\kappa}}\phi_{u_{\iota\kappa}, \bar{e}\bar{f}} \cdot b\phi_{u_\kappa, u_{\iota\kappa}}\phi_{u_{\iota\kappa}, \bar{e}\bar{f}} \\
&= (a\phi_{u_\iota, u_{\iota\kappa}}, \bar{e}\bar{f})(b\phi_{u_\kappa, u_{\iota\kappa}}, \bar{e}\bar{f}) \\
&= (a\phi_{u_\iota, u_{\iota\kappa}} \cdot b\phi_{u_\kappa, u_{\iota\kappa}}, \bar{e}\bar{f})\ .
\end{aligned}$$

This is just (1.3), which concludes the proof of Theorem 5.1.

6. The greatest idempotent-separating congruence on an orthogroup

For any semigroup S, let μ_S denote the greatest idempotent-separating congruence on S. It is well-known that μ_S is the greatest congruence contained in $\mathcal{H}_S$. An orthodox semigroup S is called *fundamental* if μ_S is the identity relation on S.

Theorem 6.1. *Let S be an orthogroup represented as in Theorem* 1.1. *Then*

$$\mu_S = \{((a, e), (a', e)) \in S \times S \colon e \in E_\iota (\iota \in Y) \text{ and } a\psi_\iota = a'\psi_\iota\} .$$

Proof. Let μ denote the relation on S defined by the right-hand side. Clearly μ is an equivalence relation contained in $\mathcal{H}_S$. To show that it is a congruence, assume that $(a, e)\,\mu(a', e)$, and let $(b, f) \in S$. Let $(a, e) \in S_\iota$ and $(b, f) \in S_\kappa$. By (1.3),

$$(a, e)(b, f) = (ab, \overline{ef}) ,$$

$$(a', e)(b, f) = (a'b, \widetilde{ef}) ,$$

where $\bar{e}$ and $\bar{f}$ are given by (1.4a) and (1.4b) with analogous expressions for $\tilde{e}$ and $\tilde{f}$. But we see that $\bar{e} = \tilde{e}$ since $a\psi_\iota = a'\psi_\iota$ implies that

$$a^{-1}\psi_\iota = (a\psi_\iota)^{-1} = (a'\psi_\iota)^{-1} = a'^{-1}\psi_\iota ,$$

while $\bar{f} = \tilde{f}$ identically. Hence

$$(a, e)(b, f)\,\mathcal{H}_S\,(a', e)(b, f) .$$

Now ab and $a'b$ belong to $G_{\iota\kappa}$, and we are to show that $(ab)\psi_{\iota\kappa} = (a'b)\psi_{\iota\kappa}$. Now

$$(ab)\psi_{\iota\kappa} = (a\phi_{\iota,\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa})\psi_{\iota\kappa} = a\phi_{\iota,\iota\kappa}\psi_{\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa}\psi_{\iota\kappa} ,$$

$$(a'b)\psi_{\iota\kappa} = a'\phi_{\iota,\iota\kappa}\psi_{\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa}\psi_{\iota\kappa} .$$

Hence we are to show that $a\phi_{\iota,\iota\kappa}\psi_{\iota\kappa} = a'\phi_{\iota,\iota\kappa}\psi_{\iota\kappa}$. But this is evident from (1.2) and $a\psi_\iota = a'\psi_\iota$. Hence

$$(a, e)(b, f)\,\mu(a', e)(b, f) .$$

The proof that μ is also a left congruence is similar.

Now let ρ be any congruence on S contained in $\mathcal{H}_S$, and let $(a, e)\,\rho(b, f)$. Since $\rho \subseteq \mathcal{H}_S$, we get $e = f$ immediately. Assume $(a, e) \in S_\iota$ (ι in Y). Then $u_\iota(a,e)u_\iota\,\rho\, u_\iota(b,f)u_\iota$: that is, $a\rho b$. Since $\rho \mid G_\iota$ is a congruence on the group G_ι, we have $a^{-1}\rho b^{-1}$. Let $x \in E(\leqslant u_\iota)$. Then $a^{-1}xa\,\rho\, b^{-1}xb$. But $a^{-1}xa$ and $b^{-1}xb$ are idempotents, and

$\rho \subseteq \mathcal{H}_S$. Hence $a^{-1}xa = b^{-1}xb$. By (4.2), $x(a\psi_\iota) = x(b\psi_\iota)$. Since this holds for all x in $E(\leqslant u_\iota)$, $a\psi_\iota = b\psi_\iota$. Hence $\rho \subseteq \mu$, showing that $\mu = \mu_S$.

Corollary 6.2. *S is fundamental if and only if all the ψ_ι are injective.*

Corollary 6.3. *$\mathcal{H}_S$ is a congruence on S if and only if all the ψ_ι are trivial* (i.e., *$G_\iota\psi_\iota$ reduces to the identity subgroup $\{\epsilon_{u_\iota}\}$ of K_ι*).

7. A reformulation of the main theorem

In this section we define, for any band E, a semilattice of groups $\overline{K}(E)$, and use this to give a simplified form of Theorem 1.1 (Theorem 7.3).

Let E be a band. If $e\mathcal{D}f$ in E, we define a mapping $\pi(e,f)$: $E(\leqslant e) \to E(\leqslant f)$ by

(7.1) $\quad x\pi(e,f) = fxf \quad$ (all x in $E(\leqslant e)$).

In the notation of Hall [5], $\pi(e,f) = \theta_{fe,ef}$. Clearly $\pi(e,e)$ is the identity transformation ϵ_e of $E(\leqslant e)$.

Lemma 7.1. (i) *If e,f,g are $\mathcal{D}$-equivalent elements of a band E, then $\pi(e,f)\,\pi(f,g) = \pi(e,g)$.*

(ii) *$\pi(e,f)$ is an isomorphism of $E(\leqslant e)$ onto $E(\leqslant f)$ with inverse $\pi(f,e)$.*

(iii) *$\pi(e,f)$ is $\mathcal{D}$-resctricted,* i.e. *$x\mathcal{D}x\pi(e,f)$ for all x in $E(\leqslant e)$.*

Proof. (i) Since $efg = eg$ if e,f,g are elements of a rectangular band, we have, for all x in $E(\leqslant e)$,

$$x\pi(e,f)\,\pi(f,g) = gfxfg = gfexefg$$

$$= gexeg = gxg = x\pi(e,g)\,.$$

(ii) Let $x, y \in E(\leqslant e)$. $e\mathcal{D}f$ implies $efe = e$, and hence

$$(fxf)(fyf) = fxefeyf = fxeyf = fxyf\ ,$$

showing that $\pi(e,f)$ is a homomorphism. Setting $g = e$ in (i), we see that $\pi(f,e)$ is inverse to $\pi(e,f)$, so $\pi(e,f)$ is bijective.

(iii) Since $\mathcal{D}$ is a congruence on E, $e\mathcal{D}f$ implies $exe\mathcal{D}fxf$; hence $x\mathcal{D}x\pi(e,f)$ for all x in $E(\leqslant e)$.

As before, $e \succ f$ means $fef = f$, and K_e is the group of $\mathcal{D}$-restricted automorphisms of $E(\leqslant e)$. For $e \succ f$ in E, define the mapping $\chi_{e,f}$: $K_e \to K_f$ as follows. For each element α of K_e, let $\alpha\chi_{e,f}$ be the transformation of $E(\leqslant f)$ defined by

$$(7.2) \qquad x(\alpha\chi_{e,f}) = f(exe)\alpha f \quad (\text{all } x \text{ in } E(\leqslant f)).$$

In terms of the π-mappings defined by (7.1),

$$(7.3) \qquad \alpha\chi_{e,f} = \pi(f, efe)\alpha\pi((efe)\alpha, f).$$

By Lemma 7.1, $\alpha\chi_{e,f}$ is a product of $\mathcal{D}$-rectricted isomorphisms

$$E(\leqslant f) \to E(\leqslant efe) \to E(\leqslant (efe)\alpha) \to E(\leqslant f)\,,$$

and so is a $\mathcal{D}$-restricted isomorphism of $E(\leqslant f)$ onto itself; that is, $\alpha\chi_{e,f} \in K_f$.

Lemma 7.2. *For $e \succ f$ in E, the mapping $\chi_{e,f}$: $K_e \to K_f$ defined by* (7.2) *or* (7.3) *is a homomorphism of groups. Moreover, $\chi_{e,e}$ is the identity map of K_e; and if $e \succ f \succ g$ in E, then $\chi_{e,f}\chi_{f,g} = \chi_{e,f}$.*

Proof. As in section 3, the symbol $\underbrace{\qquad}$ under part of an expression indicates that this part can be omitted because of Lemma 2.1. Note also that $x \in E(\leqslant f)$ implies

$$efex = efefx = efx = ex\,,$$

and hence $(efe)(exe)(efe) = exe$.

Let $\alpha, \beta \in K_e$ and $x \in E(\leqslant f)$. Then, by (7.2),

$$\begin{aligned} x(\alpha\chi_{e,f})(\beta\chi_{e,f}) &= f[ef(exe)\alpha fe]\,\beta f \\ &= f(efe)\,\beta(exe)\alpha\beta(efe)\beta f \\ &= f\underbrace{(efe)}\beta(efe)\,\alpha\beta(exe)\,\alpha\beta(efe)\,\alpha\beta\underbrace{(efe)}\beta f \\ &= f(exe)\,\alpha\beta f = x((\alpha\beta)\chi_{e,f})\ . \end{aligned}$$

Hence $\chi_{e,f}$ is a homomorphism. That $\chi_{e,e}$ is the identity map of K_e is obvious. Let $e \succ f \succ g$ in E, $\alpha \in K_e$, and $x \in E(\leqslant g)$. Note first that $e \succ g$ and $x \leqslant g$ imply

$$x = gxg = (geg)x(geg) = gexeg\ .$$

Hence, by (7.2),

$$\begin{aligned} x(\alpha\chi_{e,f}\chi_{f,g}) &= g(fxf)(\alpha\chi_{e,f})g = gf(efxfe)\alpha fg \\ &= gf(efgexegfe)\alpha fg \end{aligned}$$

$$= gf(efge)\alpha(exe)\alpha(egfe)\alpha fg$$

$$= gf\underbrace{(efge)\alpha}(ege)\alpha(exe)\alpha(ege)\alpha\underbrace{(egfe)\alpha}fg$$

$$= g(exe)\alpha g = x(\alpha\chi_{e,g}) \ .$$

Let $E = \bigcup\{E_\iota \colon \iota \in Y\}$ be the decomposition of E into a semilattice Y of rectangular bands E_ι. Select and fix an element u_ι in E_ι for each ι in Y. Write K_ι for K_{u_ι} and $\chi_{\iota,\kappa}$ for χ_{u_ι,u_κ} ($\iota \geqslant \kappa$ in Y). Let

$$\bar{K}(E) = \bigcup\{K_\iota \colon \iota \in Y\} \ .$$

By Lemma 7.2, we can make $\bar{K}(E)$ into a semilattice of groups with connecting homomorphisms $\chi_{\iota,\kappa} \colon K_\iota \to K_\kappa$ ($\iota \geqslant \kappa$ in Y) by defining product therein by

$$\alpha\beta = \alpha\chi_{\iota,\iota\kappa} \cdot \beta\chi_{\kappa,\iota\kappa} \quad \text{if } \alpha \in K_\iota \text{ and } \beta \in K_\kappa \ . \tag{7.4}$$

If $\iota \geqslant \kappa$ in Y, and we replace e by u_ι and f by u_κ in (7.2), we obtain, for all α in K_ι and x in $E(\leqslant u_\kappa)$,

$$x(\alpha\chi_{\iota,\kappa}) = u_\kappa(u_\iota x u_\iota)\alpha u_\kappa \ . \tag{7.5}$$

Hence condition (1.2) of Theorem 1.1 is equivalent to

$$x(a\phi_{\iota,\kappa}\psi_\kappa) = x(a\psi_\iota\chi_{\iota,\kappa}) \quad (\text{all } a \text{ in } G_\iota,\ x \text{ in } E(\leqslant u_\kappa)) \ ,$$

and hence to

$$\phi_{\iota,\kappa}\psi_\kappa = \psi_\iota\chi_{\iota,\kappa} \quad (\text{all } \iota \geqslant \kappa \text{ in } Y) \ . \tag{7.6}$$

Equations (7.6) assert the commutativity of each of the following diagrams (for $\iota \geqslant \kappa$ in Y):

$$\begin{tikzcd}
G_\iota \arrow[r, "\psi_\iota"] \arrow[d, "\phi_{\iota,\kappa}"] & K_\iota \arrow[d, "\chi_{\iota,\kappa}"] \\
G_\kappa \arrow[r, "\psi_\kappa"] & K_\kappa
\end{tikzcd}$$

It was observed by Grantham [3] that (7.6) are necessary and sufficient conditions on the homomorphisms $\psi_\iota \colon G_\iota \to K_\iota$ (for given ϕ's and χ's) that the mapping ψ: $Q \to \bar{K}(E)$, defined by $a\psi = a\psi_\iota$ if $a \in G_\iota$, be a homomorphism.

We summarize the foregoing in the following reformulation of Theorem 1.1. The matter of labeling components (see the example in section 1) has been incorporated into the statement of the theorem.

Theorem 7.3. *With any band $E = \bigcup\{E_\iota : \iota \in Y\}$ we can associate a semilattice of groups $\bar{K}(E) = \bigcup\{K_\iota : \iota \in Y\}$ as defined above, such that, for any semilattice of groups Q compatible with E, every orthodox extension S of E by Q is obtained as follows from an idempotent-bijective homomorphism $\psi: Q \to \bar{K}(E)$. Denoting by 1_ι the idempotent element of Q mapped by ψ onto ϵ_ι ($\iota \in Y$), and by G_ι the maximal subgroup of Q containing 1_ι, let $\psi_\iota = \psi \mid G_\iota$, and construct S as in the last paragraph of Theorem* 1.1.

8. The greatest fundamental orthogroup $K(E)$ on a band E

For convenience, let us say that an orthogroup S is *on* a band E if $E_S \cong E$. In this section we define, for any band E, a fundamental orthogroup $K(E)$ on E such that any fundamental orthogroup on E is isomorphic to a subsemigroup of $K(E)$.

Let us apply Theorem 7.3 to the case where $Q = \bar{K}(E)$ and $\psi: Q \to \bar{K}(E)$ is the identity map. Denote by $K(E)$ the resulting orthogroup. Thus $K(E) = \bigcup\{K_\iota \times E_\iota : \iota \in Y\}$ with product defined as follows. If $\iota, \kappa \in Y$, $(\alpha, e) \in K_\iota \times E_\iota$, and $(\beta, f) \in K_\kappa \times E_\kappa$, then

$$(8.1) \qquad (\alpha, e)(\beta, f) = (\alpha\beta, \bar{e}\bar{f}),$$

where $\alpha\beta$ is given by (7.4), and

$$(8.2a) \qquad \bar{e} = e(u_\iota efeu_\iota)\alpha^{-1}e,$$

$$(8.2b) \qquad \bar{f} = f(u_\kappa fefu_\kappa)\beta f.$$

$\bar{K}(E)$ is, of course, the greatest inverse homomorphic image of $K(E)$. Denote by η the projection $(\alpha, e)\eta = \alpha$ of $K(E)$ onto $\bar{K}(E)$.

The mapping $e \mapsto \hat{e} = (\epsilon_{u_\iota}, e)$, where $e \in E_\iota$, is an isomorphism of E onto the band $\hat{E} = E_{K(E)}$ of idempotents of $K(E)$. A subsemigroup of $K(E)$ is called *full* if it contains $\hat{E}$.

Theorem 8.1. *If S is an orthogroup expressed via Theorem 7.3 as an orthodox extension of E by Q, and thus determined by a homomorphism $\psi: Q \to \bar{K}(E)$, then the mapping $\theta: S \to K(E)$ defined by*

$$(8.3) \qquad (a, e)\theta = (a\psi, e) \quad (\textit{for all } (a, e) \textit{ in } S)$$

is a homomorphism mapping E_S isomorphically onto $\hat{E}$, the kernel of which is the greatest idempotent-separating congruence μ_S on S. In particular, K(E) is fundamental, and every fundamental orthogroup on E is isomorphic to a full subsemigroup of K(E).

Proof. In accordance with Theorem 7.3, if $Q = \bigcup \{G_\iota : \iota \in Y\}$ then $S = \bigcup \{G_\iota \times E_\iota : \iota \in Y\}$, with product given by (1.3) and (1.4). If $(a, e) \in G_\iota \times E_\iota$, then, by (8.3), $(a, e)\theta = (a\psi_\iota, e)$ since $\psi_\iota = \psi \mid G_\iota$.

Let $(a, e) \in G_\iota \times E_\iota$ and $(b, f) \in G_\kappa \times E_\kappa$. By (1.3) and (8.3),

$$[(\alpha, e)(b, f)]\,\theta = (a\phi_{\iota,\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa}, \overline{e}\overline{f})\theta$$

$$= (a\phi_{\iota,\iota\kappa}\psi_{\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa}\psi_{\iota\kappa}, \overline{e}\overline{f}),$$

with $\overline{e}, \overline{f}$ given by (1.4). By (8.3), (8.1) and (7.4),

$$(a, e)\theta \cdot (b, f)\theta = (a\psi_\iota, e)(b\psi_\kappa, f)$$

$$= (a\psi_\iota\chi_{\iota,\iota\kappa} \cdot b\psi_\kappa\chi_{\kappa,\iota\kappa}, \overline{e}\overline{f}),$$

with the same $\overline{e}$ and $\overline{f}$, since α and β in (8.2) are now to be replaced by $a\psi_\iota$ and $b\psi_\kappa$, respectively. The first components are equal by (7.6), and we conclude that θ is a homomorphism.

The idempotents of S are the elements $(1_\iota, e)$ with e in E_ι, and $(1_\iota, e)\theta = (\epsilon_{u_\iota}, e) = \hat{e}$, so θ induces an isomorphism of E_S onto $\hat{E}$.

Let $(a, e) \in G_\iota \times E_\iota$ and $(b, f) \in G_\kappa \times E_\kappa$. If $(a, e)\theta = (b, f)\theta$, then $(a\psi_\iota, e) = (b\psi_\kappa, f)$, so $e = f$ and $\iota = \kappa$, and $a\psi_\iota = b\psi_\iota$. We conclude from Theorem 6.1 that $(a, e)\mu_S (b, f)$. Since the argument can be reversed, we see that the kernel of θ is μ_S. The final statement of the theorem is now evident.

From the last assertion of Theorem 8.1 it follows that $K(E)$ is the same as that in Hall's theorem stated in the introduction. We can now prove Hall's theorem by showing that the semigroup S constructed in Theorem 7.3 from an idempotent-bijective homomorphism $\psi: Q \to \overline{K}(E)$ is isomorphic with the spined product P of Q and $K(E)$ with respect to ψ and η: $K(E) \to \overline{K}(E)$.

P consists of all elements $((\alpha, e), a)$ of the direct product $K(E) \times Q$ such that $(\alpha, e)\eta = a\psi$; that is, such that $a\psi = \alpha$. If $a \in G_\iota$, then $\alpha = a\psi_\iota \in K_\iota$ and $e \in E_\iota$, since $(\alpha, e) \in K(E)$. Hence the mapping $\zeta: S \to P$ defined by

$$(a, e)\zeta = ((a\psi, e), a)$$

is bijective. Moreover, it is an isomorphism. For suppose $(a, e) \in G_\iota \times E_\iota$ and

$(b,f) \in G_\kappa \times E_\kappa$. Then, by (8.1), (7.4), (7.6), (1.1), and (1.3),

$$
\begin{aligned}
(a,e)\zeta \cdot (b,f)\zeta &= ((a\psi_\iota, e), a)((b\psi_\kappa, f), b) \\
&= ((a\psi_\iota, e)(b\psi_\kappa, f), ab) \\
&= ((a\psi_\iota \chi_{\iota,\iota\kappa} \cdot b\psi_\kappa \chi_{\kappa,\iota\kappa}, \bar{e}\bar{f}), ab) \\
&= ((a\phi_{\iota,\iota\kappa}\psi_{\iota\kappa} \cdot b\phi_{\kappa,\iota\kappa}\psi_{\iota\kappa}, \bar{e}\bar{f}), ab) \\
&= (((ab)\psi_{\iota\kappa}, \bar{e}\bar{f}), ab) \\
&= (ab, \bar{e}\bar{f})\zeta = [(a,e)(b,f)]\zeta \ ,
\end{aligned}
$$

noting that $\bar{e}$ and $\bar{f}$ as given by (8.2) are identical with those needed for (1.4) in the last step.

We may of course consider S itself as the spined product. When we do this, the two unlabelled arrows in the pull-back diagram

$$
\begin{tikzcd}
S \arrow[r] \arrow[d] & Q \arrow[d, "\psi"] \\
K(E) \arrow[r, "\eta"'] & \bar{K}(E)
\end{tikzcd}
$$

are as follows: $S \to Q$ is the projection $(a,e) \to a$, and $S \to K(E)$ is given by $(a,e) \to (a\psi, e)$.

Our final theorem supplements Hall's theorem by giving explicit descriptions of $K(E)$ and $\bar{K}(E)$. It is, of course, immediate from the definitions we have given of these semigroups, and Theorem 8.1.

Theorem 8.2. *Let $E = \bigcup \{E_\iota : \iota \in Y\}$ be a band. The greatest fundamental orthogroup $K(E)$ on E, and its greatest inverse homomorphic image $\bar{K}(E)$, have the following structure. Select a representative element u_ι from each E_ι $(\iota \in Y)$, and let K_ι be the $\mathcal{D}$-restricted automorphism group of the band $E(\leqslant u_\iota)$. Then $\bar{K}(E)$ is isomorphic with the semilattice of groups $\bigcup \{K_\iota : \iota \in Y\}$ having connecting homomorphisms $\chi_{\iota,\kappa}$ $(\iota \geqslant \kappa)$ defined by* (7.5), *and $K(E)$ is isomorphic with the orthogroup $\{K_\iota \times E_\iota : \iota \in Y\}$ with product defined by* (8.1), (8.2) *and* (7.4).

9. Splitting orthogroups

If $E = \bigcup \{E_\iota : \iota \in Y\}$ is a band expressed as a semilattice Y of rectangular bands E_ι, and $Q = \bigcup \{G_\iota : \iota \in Y\}$ is a semilattice of groups, with the same Y, then we may form the spined product P (see section 8) of E and Q with respect to the natural homomorphisms $E \to Y$ and $Q \to Y$. Thus $P = \bigcup \{G_\iota \times E_\iota \colon \iota \in Y\}$ with product defined by

$$(9.1) \qquad (a, e)(b, f) = (ab, ef) \quad (\text{all } (a, e), (b, f) \text{ in } P).$$

We say that an orthogroup S *splits* if it is the spined product of E_S and Q_S.

Yamada [12, 13] showed the equivalence of (i) and (ii) in the next theorem.

Theorem 9.1. *For an orthogroup S, the following are equivalent.*

(i) *S splits.*

(ii) *Green's relation $\mathcal{H}$ on S is a congruence (that is, S is a band of groups).*

(iii) *In some (or every) representation of S as in Theorem 1.1 or 7.3, the homomorphisms $\psi_\iota \colon G_\iota \to K_\iota$ are all trivial (that is, $G_\iota \psi_\iota = \{\epsilon_{u_\iota}\}$ for every ι in Y).*

Proof. That (i) implies (ii) is immediate from the description of P in the first paragraph of this section, since clearly $(a, e)\mathcal{H}(b, f)$ if and only if $e = f$.

Assuming (ii), we have $\mu_S = \mathcal{H}$, and we conclude from Theorem 6.1 that $a\psi_\iota = a'\psi_\iota$ for all a, a' in G_ι (for each ι in Y), hence that $a\psi_\iota = 1_\iota \psi_\iota = \epsilon_{u_\iota}$. Thus (iii) holds, using the word "every".

Assuming (iii), with the word "some", we see from (1.4) that $\bar{e} = efe$ and $\bar{f} = fef$, so (1.3) reduces to (9.1). Hence (i) holds.

Another interesting result of Yamada [12, 13] is that, for a normal band E, every orthogroup on E splits. Our concluding theorem specifies just what bands E have this property.

Theorem 9.2. *The following conditions on a band E are equivalent.*

(i) *Every orthogroup on E splits.*

(ii) *$K(E)$ is a band (that is, $K(E) = \hat{E}$).*

(iii) *For each e in E, the only $\mathcal{D}$-restricted automorphism of $E(\leqslant e)$ is the identity map.*

Proof. Assume (i). Then, in particular, $K(E)$ splits, so $\mathcal{H}$ is a congruence on $K(E)$, by Theorem 9.1. But $K(E)$ is fundamental, by Theorem 8.1. Hence $\mathcal{H}$ is the equality relation on $K(E)$, which implies that $K(E)$ is a band.

Assuming (ii), each of the maximal subgroups K_e of $K(E)$ is trivial; that is, (iii) holds.

Assume (iii), and let S be an orthogroup on E. On representing S as in Theorem 1.1, each ψ_ι is trivial, since each K_ι is trivial by (iii). By Theorem 9.1, S splits, and hence (i) holds.

Yamada's theorem is an immediate consequence of this. For a band E is normal if and only if it has the following property: if $\iota > \kappa$ in Y, and $e \in E_\iota$, there exists exactly one element f of E_κ such that $e > f$. This follows by combining the results of Yamada and Kimura [15] with those of Howie [6]; see also Petrich [9; IV.4]. But this property evidently implies (iii).

There are plenty of bands satisfying (iii) which are not normal. For example, let $E = \{1, e, f, g\}$ with table as shown. Y is a chain, $\iota > \kappa > \lambda$, with $E_\iota = \{1\}$,

	e	f	g
e	e	f	g
f	f	f	g
g	f	f	g

$E_\kappa = \{e\}$, and $E_\lambda = \{f, g\}$. Since $1 > f$ and $1 > g$, E is not normal. But it is easily checked that E has no non-trivial automorphisms.

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